Close Tue: 14.3(2), 14.4
Close Thur: 14.7
Reminder of Extra Help Options
Office Hours - 12:30-2:00pm (PDL C-339)
Math Tutors - 9:30am–9:30pm (Com B-014)
CLUE Tutors - 7:00pm – midnight (Mary Gates)

14.4/14.7 Tangent Planes & Max/Min

Remember from Math 124:
The equation of the tangent line to the curve $y = f(x)$ at x_0 is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Similarly, the equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0) is given by

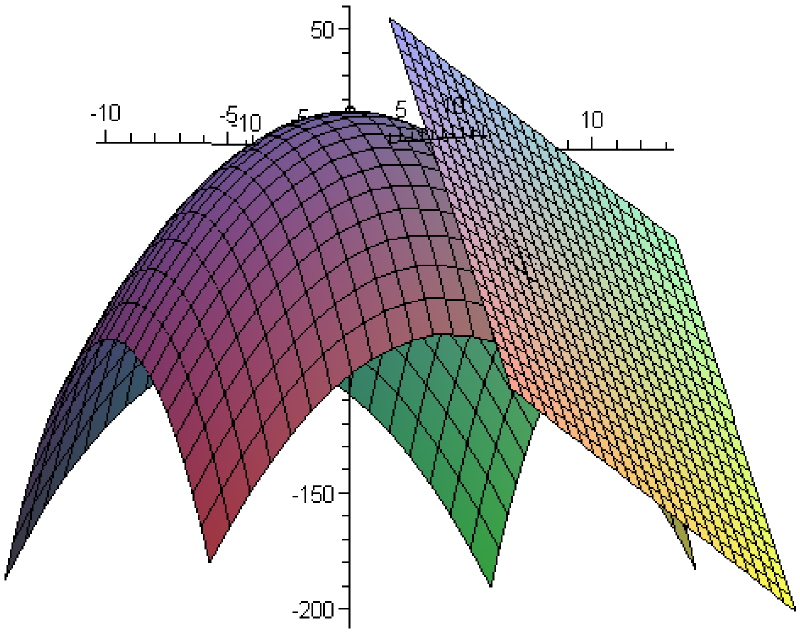
$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Entry Tasks:

1. Find the equation for the tangent line to $f(x) = x^3$ at $x = 2$.

2. Find the equation for the tangent plane to $f(x, y) = x^2 + 3y^2x - y^3$ at $(x, y) = (2, 1)$.

Visuals and Derivation of Tangent Plane



$z = f(x,y) = 15 - x^2 - y^2$ at (7,4)

Recall:

$f(7,4) = -50$

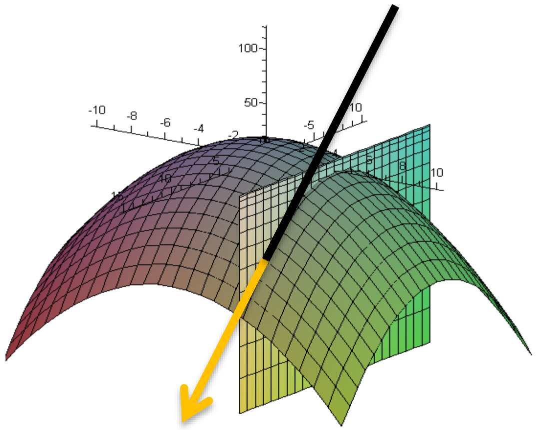
$f_x(7,4) = -14$

$f_y(7,4) = -8$

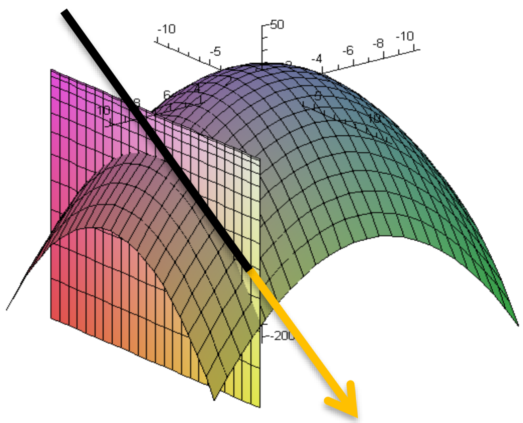
Derivation of Tangent Plane

The plane goes thru (7, 4, -50).
Now we need a normal vector.

Note:
$f_x(x,y) = -2x$
$f_x(7,4) = -14$



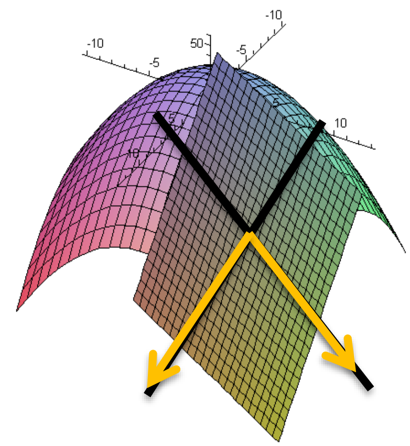
$f_y(x,y) = -2y$
$f_y(7,4) = -8$



Thus, we can get two vectors that are parallel to the plane:

$$<1,0,f_x(x_0,y_0)> = <1,0,-14>$$
$$<0,1,f_y(x_0,y_0)> = <0,1,-8>$$

So a normal vector is given by
$<1,0,-14> \times <0,1,-8> = <14, 8, 1>$



Tangent Plane:

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

General Derivation

In general, for $z = f(x,y)$ at (x_0, y_0) by:

1. $z_0 = f(x_0, y_0)$ = height.

2. $\langle 1,0, f_x(x_0, y_0)\rangle =$ 'a tangent in *x*-dir.'
   $\langle 0,1, f_y(x_0, y_0)\rangle =$ 'a tangent in *y*-dir.'

3. Normal to surface:
   $$\langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle$$
   $$= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear Approximation: "Near" the point (x_0,y_0) the tangent plane and surface have z-values that are close together.

Take $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$,
add z_0 to both sides to get

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We call the above function, $L(x, y)$, the linear approximation to $f(x, y)$ at (x_0, y_0).

Idea: $f(x, y) \approx L(x, y) \text{ for } (x, y) \approx (x_0, y_0)$

Example: Use the linear approximation to $f(x, y) = x^2 + 3y^2x - y^3$ at $(x, y) = (2, 1)$ to estimate the value of $f(1.9, 1.05)$.

14.7 Local Max/Min

Consider the surface z = f(x,y).

Some Terminology:

A **local maximum** occurs at (a,b) if f(a,b) is larger than *all* values "near" it (top of a hill).

A **local minimum** occurs at (a,b) if f(a,b) is smaller than *all* values "near" it (bottom of a valley).

A **critical point** is a point (a,b) where <u>**BOTH**</u>

$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

or where either partial doesn't exist.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, and (a,b) is not a local max or min, then we call it a **saddle point**.

Example: Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

Second Derivative Test

Let (a,b) be a critical point.

Find all **second** partials at (a,b)

and compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

1. If **D > 0**, then the concavity is the same in all directions. So

   (a) If $\mathbf{f_{xx} > 0}$, then it is concave up in all directions. **Local Minimum.**

   (b) If $\mathbf{f_{xx} < 0}$, then it is concave down in all directions. **Local Maximum.**

2. If **D < 0**, then the concavity changes in some direction. **Saddle Point**.

3. If **D = 0**, the test is **inconclusive** .
   (need a contour map)

Quick Examples:

1. $f(x,y) = 15 - x^2 - y^2$,

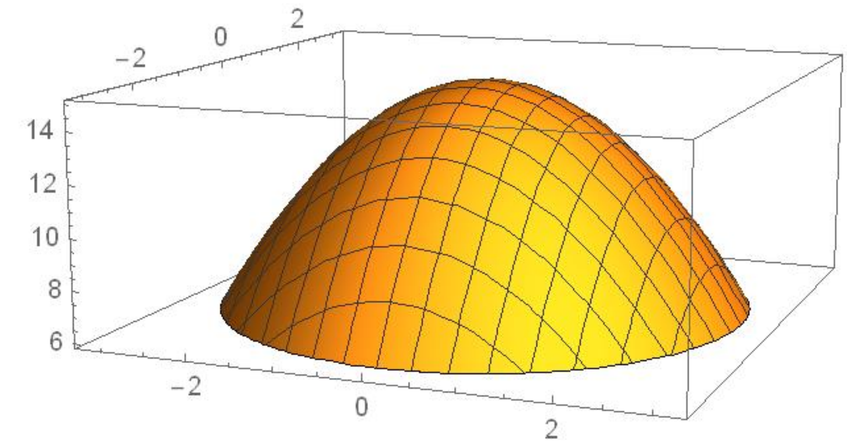
Critical pt: (0,0).

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$

$D = (-2)(-2)-(0)^2 = 4$

$D > 0$, $f_{xx} < 0$, $f_{yy} < 0$

Local max!



2. $f(x,y) = x^2 + y^2$,

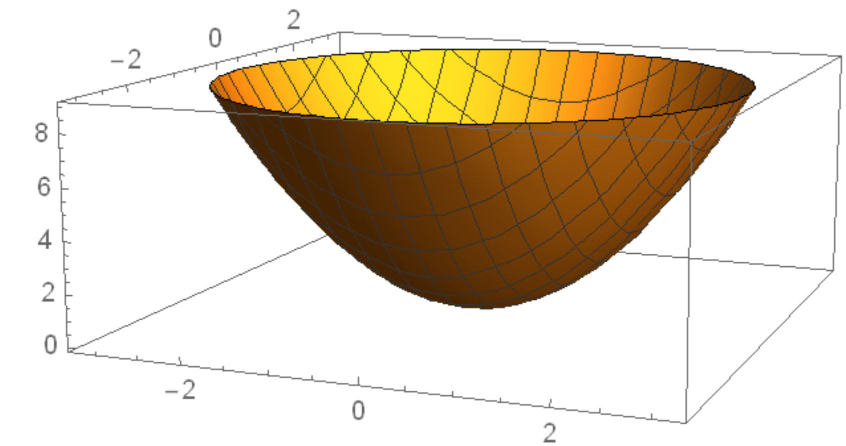
Critical pt: (0,0).

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$,

$D = (2)(2)-(0)^2 = 4$

$D > 0$, $f_{xx} > 0$, $f_{yy} > 0$

Local min!



3. $f(x,y) = x^2 - y^2$

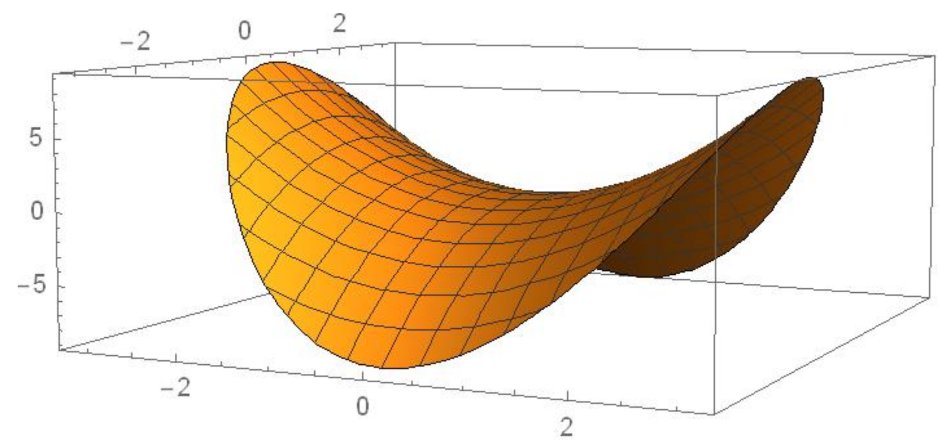
Critical pt: (0,0).

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$,

$D = (2)(-2)-(0)^2 = -4$

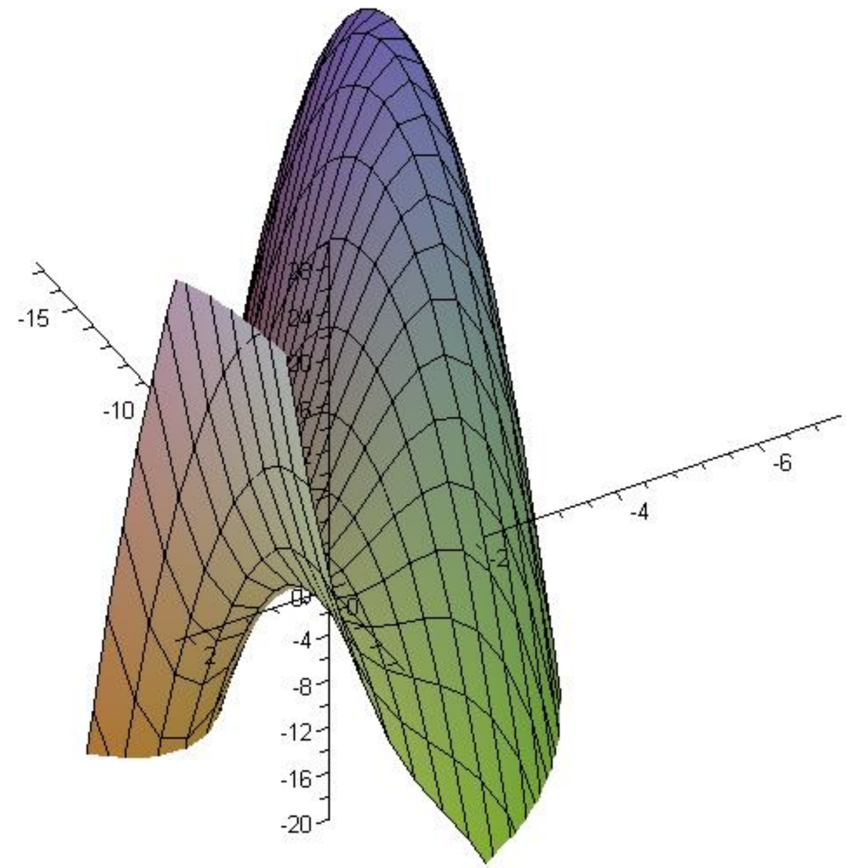
$D < 0$ (note also, $f_{xx} < 0$, $f_{yy} > 0$)

Saddle point!



Example: Find and classify all critical points for

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$



Examples from old exams:

1. Find and classify all critical points for

$$f(x,y) = x^2 + 4y - x^2y + 1$$

2. Find and classify all critical points for

$$f(x,y) = \frac{9}{x} + 3xy - y^2$$

3. Find and classify all critical points for

$$f(x,y) = x^2y - 9y - xy^2 + y^3$$